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GROUPS OF PRIME POWER ORDER AS FROBENIUS-WIELANDT COMPLEMENTS

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ABSTRACT. It is known that the Sylow subgroups of a Frobenius complement are cyclic or generalized quaternion. In this paper it is shown that there are no restrictions at all on the structure of the Sylow subgroups of the Frobenius-Wielandt complements that appear in the well-known Wielandt's generalization of Frobenius' Theorem. Some examples of explicit constructions are also given.

0. INTRODUCTION

Let G be a finite group acting on a complex vector space M. As in [LP], let $N(G, M)$ be the (normal) subgroup of G generated by those elements of G that fix a nontrivial vector in M. Let N be any normal subgroup of G containing $N(G, M)$. The factor group G/N will be called a *Frobenius-Wielandt complement* (or, shortly, an FW-*complement*) for G. For an explanation of this name see [E], where these factor groups are shown to be exactly those appearing in Wielandt's generalization of Frobenius' Theorem.

In the particular case $N = N(G, M) = 1$ it is well known that the Sylow p-subgroups of G/N are cyclic or generalized quaternion.

On the other hand, one can ask if, given arbitrarily a p-group X, there exists an FW-complement G/N isomorphic to X: here we have the following

Theorem. *Let X be a finite p-group. Then X is a Frobenius-Wielandt complement: there exists a finite p-group G with a normal subgroup N and a complex G-module M such that $N \supseteq N(G, M)$, and G/N is isomorphic to X.*

§1 is devoted to the proof of the above Theorem, but also contains some results about the p-dimension subgroups of a free group, which turn out to be useful in our setting. Some of these results are known, or at least they belong to the folklore of the theory (e.g., Lemma 1.9, in which the techniques are the same as in [HB, VIII.11.8(c)], or Lemmas 1.11, 1.12 that are essentially contained in [Z, W2]); they are proved here for the convenience of the reader.

We note here that the proof of the Theorem rests on the fact that the definition of an FW-complement above is closed under taking factor groups; in other words, homomorphic images of FW-complements are FW-complements.

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In fact, we prove here that every group in a particular class of finite p-groups can be exhibited as an FW-complement, and then that every finite p-group is isomorphic to a homomorphic image of a group in that class.

However, it is clear that the particular case $N = N(G, M)$ is the most interesting, because of its intrinsic representation theoretical meaning; actually, the whole research is motivated by this case. But the techniques used in §1 do not allow us to specify easily the structure of $G/N(G, M)$, once we have constructed G and M, beyond the fact that X is isomorphic to a homomorphic image of it. In general, it seems to be quite hard to decide if a given p-group is isomorphic to a factor group of the form $G/N(G, M)$, for suitable G and M.

Thus, in §2, we specialize our techniques to construct explicitly some examples of nonabelian FW-complements for which $N = N(G, M)$; this generalizes the construction given in [S2] for $p = 3$ to all odd primes.

Results and examples in the setting of FW-complements always yield corollaries in the context of the so-called Hughes Problem. This connection is explained in detail in [LP, S1]; here we just note that the Theorem above can be used to construct groups G with a "large" Hughes factor group $G/H_{p^n}(G)$.

The notation is standard. We indicate by G_i the ith term of the lower central series of a group G, and by G^n the subgroup generated by the nth powers of elements of G.

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The following result shows that our problem can be stated in purely group-theoretical terms:

1.1 **Lemma** [S2]. *Let G be a finite p-group and N a normal subgroup of G. Then the following are equivalent*:

(i) *There exists a complex G-module M such that N contains every element of G that fixes a nontrivial vector in M.*

(ii) *There is a cyclic section H/K of G such that every element of $G - N$ has a power in $H - K$.*

Proof. Suppose first that G has a module M as required. We may assume that M is irreducible: in fact, if V is an irreducible component of M, $N(G, V) \leq N(G, M)$. Then, since G is a p-group, $M = L^G$, where L is a linear module for some subgroup H of G, and let K be the kernel of the action of H on L. Suppose $x \in G - N$, and say that μ is the character afforded by M, and λ the character afforded by L. We have

$$0 = (\mu_{\langle x\rangle}, 1_{\langle x\rangle}).$$

Now, by Mackey's theorem and Frobenius reciprocity,

$$(\mu_{\langle x\rangle}, 1_{\langle x\rangle}) = (\lambda^G_{\langle x\rangle}, 1_{\langle x\rangle}) = \left(\sum_y (\lambda^y_{H^y\cap\langle x\rangle})^{\langle x\rangle}, 1_{\langle x\rangle}\right)$$
$$= \sum_y ((\lambda^y_{H^y\cap\langle x\rangle})^{\langle x\rangle}, 1_{\langle x\rangle}) = \sum_y (\lambda^y_{H^y\cap\langle x\rangle}, 1_{H^y\cap\langle x\rangle}),$$

where y runs over a set of double cosets representatives of H and $\langle x\rangle$. In particular,

$$(\lambda_{H\cap\langle x\rangle}, 1_{H\cap\langle x\rangle}) = 0,$$

and therefore $H\cap\langle x\rangle$ is not contained in K. On the other hand, assume H/K is a section as required. Induce to G a linear character λ of H with kernel K. Set $\mu = \lambda^G$. By the normality of N, $H^g\cap\langle x\rangle$ properly contains $K^g\cap\langle x\rangle$ for all $g\in G$; reading the equalities above backwards, we get the result. □

The following lemma generalizes slightly the argument in the proof of [HB, Proposition VIII.1.12].

1.2 **Lemma.** *Let G be a group and, for $i = 1, \dots, s$, let $n_i \geq 1$ and $k_i \geq 0$ be integers. Let p be a prime. Then*

$$[G^{p^{k_1}}_{n_1}, G^{p^{k_2}}_{n_2}, \dots, G^{p^{k_s}}_{n_s}] \leq H(n_i, k_i) = \prod_{\substack{(h_1,\dots,h_s)\\ 0\leq h_i\leq k_i}} G^{p^{k-h}}_{\Sigma},$$

where

$$h = \sum_{i=1}^{s} h_i, \quad k = \sum_{i=1}^{s} k_i, \quad and \quad \Sigma = \sum_{i=1}^{s} n_i p^{h_i}.$$

Proof. For $s = 1$, the result is trivial. For $s = 2$, as in [HB, Proposition VIII.1.12], it is enough to show

$$[a^{p^{k_1}}, b^{p^{k_2}}] \leq H(n_1, n_2, k_1, k_2)$$

for arbitrary $a\in G_{n_1}$ and $b \in G_{n_2}$. Set $c = [a, b^{p^{k_2}}]$ and $D = \langle a, c\rangle$. By [HB, Proposition VIII.1.1(b)],

$$(1) \qquad [a^{p^{k_1}}, b^{p^{k_2}}] \equiv c^{p^{k_1}} \mod D_2^{p^{k_1}} \prod_{i=1}^{k_1} (D_{p^i})^{p^{k_1-i}}.$$

Define, for $0\leq i\leq k_1$,

$$E(i) = \prod_{j=0}^{k_2} (G_{n_1p^i+n_2p^j})^{p^{k_2-j}}.$$

By [HB, Proposition VIII.1.2],

$$(2) \qquad c\in E(0).$$

Thus, $D \leq \langle a, E(0)\rangle$. Then by [HB, Proposition VIII.1.11], $D_{p^i} \leq E(i)$. Therefore (1) and (2) give

$$[a^{p^{k_1}}, b^{p^{k_2}}] \in \prod_{i=0}^{k_1} E(i)^{p^{k_1-i}} = H(n_1, n_2, k_1, k_2). \tag{3}$$

The general case is then reduced to the case $s = 2$ as follows. Assume, by induction, the result is true for $s-1$. Then it is enough to show, for $H = H(n_1, \ldots, n_s, k_1, \ldots, k_s)$,

$$[H(n_1, \ldots, n_{s-1}, k_1, \ldots, k_{s-1}), G_{n_s}^{p^{k_s}}] \leq H.$$

Therefore it suffices to show, for any choice of $(h_1, h_2, \ldots, h_{s-1})$ such that $0 \leq h_i \leq k_i$, that we have

$$[(G_{\Sigma'})^{p^{\sigma'}}, G_{n_s}^{p^{k_s}}] \leq H,$$

where

$$\Sigma' = \sum_{j=1}^{s-1} n_j p^{h_j}, \qquad \sigma' = \sum_{j=1}^{s-1}(k_j - h_j).$$

Suppose now $0 \leq l \leq \sigma'$, and let $\overline{h}_i$ $(i = 1, 2, \ldots, s-1)$ be integers such that $0 \leq h_i \leq \overline{h}_i \leq k_i$ and $\sum_{j=1}^{s-1}(k_j - \overline{h}_j) = \sigma' - l$. Thus, we have

$$l = \sigma' - \left(\sum_{j=1}^{s-1} k_j - \overline{h}_j\right) = \sum_{j=1}^{s-1}(\overline{h}_j - h_j) \geq \overline{h}_i - h_i \geq 0$$

for each i, and then

$$p^k \Sigma' = \sum_{j=1}^{s-1} n_j p^{h_j + l} \geq \sum_{j=1}^{s-1} n_j p^{\overline{h}_j}. \tag{4}$$

By (3),

$$[G_{\Sigma'}^{p^{\sigma'}}, G_{n_s}^{p^{k_s}}] \leq \prod_{\substack{(l, h_s) \\ 0 \leq l \leq \sigma' \\ 0 \leq h_s \leq k_s}} (G_{p^l\Sigma' + n_s p^{h_s}})^{p^{\sigma' + k_s - l - h_s}};$$

therefore it suffices to show

$$(G_{p^l\Sigma' + n_s p^{h_s}})^{p^{\sigma' + k_s - l - h_s}} \leq H.$$

But by (4)

$$(G_{p^l\Sigma' + n_s p^{h_s}})^{p^{\sigma' + k_s - l - h_s}} \leq (G_{\sum_{j=1}^{s-1} n_j p^{\overline{h}_j} + n_s p^{h_s}})^{p^{(\sum_{j=1}^{s-1} k_j - \overline{h}_j) + k_s - h_s}} \leq H. \quad \square$$

1.3 **Definition.** Let G be a group and p a prime. For any positive integer n, write

$$\kappa_n(G) = \prod_{ip^k \geq n} G_i^{p^k}.$$

$\kappa_n(G)$ is a characteristic subgroup of G, and $G = \kappa_1(G) \geq \kappa_2(G) \geq \cdots$ is called the Jennings-Lazard-Zassenhaus series of G, or the p-lower central series of G. For the elementary properties of $\kappa_n(G)$, the reader is referred to [HB, §VIII.1].

In what follows, if n, m are positive integers, by $\lceil n/m \rceil$ is denoted the *smallest* positive integer *larger* than n/m. From 1.2 follows:

1.4 **Corollary.** *Assume the notation as in* 1.2. *Set* $r = \sum_{i=1}^{s} n_i p^{k_i}$ *and* $m = \min_{1\leq i\leq s, k_i>0}(\hat{m}_i + n_i p^{k_i-1})$, *where* $\hat{m}_i = \sum_{j\neq i} n_j p^{k_j}$. *Then*

$$[G_{n_1}^{p^{k_1}}, G_{n_2}^{p^{k_2}}, \ldots, G_{n_s}^{p^{k_s}}] \leq G_r \kappa_{pm}(G).$$

Proof. For all $n \geq 1$ and $l \geq 0$, we have $G_n \leq \kappa_n(G)$ and $G_n^{p^l} \leq \kappa_n(G)^{p^l} \leq \kappa_{p^l n}(G)$, by [HB, Proposition VIII.1.13]. By 1.2, $[G_{n_1}^{p^{k_1}}, G_{n_2}^{p^{k_2}}, \ldots, G_{n_s}^{p^{k_s}}]$ is contained in a product of subgroups of the form $G_\Sigma^{p^{k-h}}$ for suitable choices of h, k, Σ. Then the corollary will be completely proved if we show that, in the notation of 1.2, we have $p^{k-h}\Sigma \geq pm$, whenever $k > h$. Now if $k > h$, then $k_i > h_i$ for some i. Set $b_j = k_j - h_j$ if $j \neq i$, and $b_i = k_1 - h_i - 1$. Now

$$p^{k-h}\Sigma = p(p^{\sum_{j=1}^{s} b_j}) \left(\sum_{j=1}^{s} n_j p^{h_j}\right) \geq p \left(\sum_{j=1}^{s} n_j p^{h_j+b_j}\right) = pm. \quad \square$$

1.5 **Lemma.** *Let* G *be a group,* p *a prime, and let* n, m_i *be positive integers for* $i = 1, \ldots, s$. *Set* $\kappa_{m_i} = \kappa_{m_i}(G)$, *and* $r = \sum_{i=1}^{s} m_i$, $\hat{m}_j = \sum_{i\neq j} m_i$. *Assume* $\hat{m}_j + \lceil m_j/p \rceil \geq \lceil n/p \rceil$ *for all* j. *Then*

$$[\kappa_{m_1}, \ldots, \kappa_{m_s}] \leq G_r \kappa_n(G).$$

Proof. By the definition of $\kappa_{m_i}(G)$, it is enough to show that, whenever $n_i p^{k_i} \geq m_i$,

$$[G_{n_1}^{p^{k_1}}, G_{n_2}^{p^{k_2}}, \ldots, G_{n_s}^{p^{k_s}}] \leq G_r \kappa_n(G).$$

Note that, if $k_i > 0$ for some i,

$$p\left(\sum_{j\neq i} n_j p^{k_j} + n_i p^{k_i-1}\right) \geq p\left(\sum_{j\neq i} m_j + \left\lceil \frac{m_i}{p} \right\rceil\right) \geq p\left\lceil \frac{n}{p} \right\rceil \geq n,$$

and that

$$\sum_{i=1}^{s} m_i \leq \sum_{i=1}^{s} n_i p^{k_i}.$$

Thus 1.4 gives the result. $\square$

Throughout the rest of this section, $F = \langle x_1, \dots, x_d \rangle$ will be the free group on d generators; $\mathscr{K}$ will be a field of prime characteristic p; and $\kappa_l = \kappa_l(F)$.

It is well known [HB, §VIII.1] that κ_l/κ_{l+1} is a finite elementary abelian p-group, so that F/κ_{l+1} is a finite p-group. Set $p^{d_l} = [\kappa_l : \kappa_{l+1}]$.

1.6 **Lemma** [HB, Exercise VIII.3, p. 265]. *Let $\mathscr{A}_q$ be the $\mathscr{K}$-algebra having $\mathscr{K}$-basis $\{a_{i_1} \cdots a_{i_n} | 0 \le n \le q\}$ for elements $a_1, \dots, a_d$, in which any product of $q+1$ of the a_i is 0. Then $1 + a_i$ is a unit of $\mathscr{A}_q$, for all i, and, for $G = \langle 1 + a_i, i \in \{1 \cdots d\} \rangle$, we have*

- (i) *$\mathscr{A}_q \simeq \mathscr{K}F/\mathscr{I}^{q+1}$, where F is the free group on d generators and $\mathscr{I}$ is the augmentation ideal of $\mathscr{K}F$.*
- (ii) *$G \simeq F/\kappa_{q+1}$.*

Sketch of proof. A map $\theta : \mathscr{K}F \to \mathscr{A}_q$ is defined naturally: $\theta(x_i) = 1 + a_i$. Then θ is an epimorphism, its kernel is $\mathscr{I}^{q+1}$, and this gives (i), while Jennings' Theorem [HB, Proposition VIII.2.7] gives (ii). □

We summarize in the next statements some results of [W].

1.7 **Theorem.** *Let G be as in 1.6, where $\mathscr{K} = GF(p)$ and $q = p^n$, n a positive integer. Assume $q \ge d$. Then there exists a maximal subgroup M of G^q such that each element of $G - \Phi(G)$ has its qth power in $M - G^q$.*

Let π_l be the projection $\pi_l : F \to F/\kappa_{l+1}$. By [H, Proposition III.18.2], π_l induces a group-algebra epimorphism

$$\pi_l^* : \mathscr{K}F \to \mathscr{K}(F/\kappa_{l+1})$$

whose kernel is generated by elements of the form $(k-1)x$, where $k \in \operatorname{Ker} \pi_l = \kappa_{l+1}$ and $x \in F$.

By Jennings' Theorem [HB, Proposition VIII.2.7] $\ker \pi_l^* \subseteq \mathscr{I}^{l+1}$, where $\mathscr{I}$ is the augmentation ideal of $\mathscr{K}F$. Furthermore, $\pi_l^*(\mathscr{I}^j) = \overline{\mathscr{I}}^j$ for $j \ge 1$, where $\overline{\mathscr{I}}$ is the augmentation ideal of $\mathscr{K}(F/\kappa_{l+1})$. Let θ be as in the proof of 1.6. It is clear that $\theta = \overline{\theta} \circ \pi_l^*$, where $\overline{\theta} : \mathscr{K}(F/\kappa_{l+1}) \to \mathscr{A}_l$ is an epimorphism and $\ker \overline{\theta} = \overline{\mathscr{I}}^{l+1}$. Furthermore, for $\mathscr{B}^j = \langle a_{i_1} \cdots a_{i_j} | i_l \in \{1 \cdots d\} \rangle \mathscr{A}_l$, we have $\overline{\theta}(\overline{\mathscr{I}}^j) = \mathscr{B}^j$ for all $j \ge 1$. It is easy to compute the dimension over $\mathscr{K}$ of $\mathscr{B}^j/\mathscr{B}^{j+1}$: it is equal to the number of different monomials of degree j in d noncommutative variables; that is, d^j if $1 \le j \le l$. Therefore, with the notation above:

1.8 **Lemma.** *For $0 \le j \le l$, $\dim_{\mathscr{K}} \overline{\mathscr{I}}^j/\overline{\mathscr{I}}^{j+1} = d^j$.*

It is now possible to compute an inductive formula for the dimension d_n of the p-lower factor κ_n/κ_{n+1}.

1.9 Lemma. *Set*

$$d_{i/p} = \begin{cases} d_r & \text{if } i = pr, \\ 0 & \text{if } p \nmid i. \end{cases}$$

Then

$$d_n = \frac{1}{n} \sum_{k|n} \mu(k) d^{n/k} + d_{n/p},$$

where μ is the Möbius function.

Proof. F/κ_{l+1} is a p-group, and therefore Jennings' formula [HB, Proposition VIII.2.10] can be applied: since

$$p^{d_i} = [\kappa_i(F) : \kappa_{i+1}(F)] = [\kappa_i(F/\kappa_{l+1}) : \kappa_{i+1}(F/\kappa_{l+1})]$$

for $i \leq l$,

$$\prod_{i=1}^{l} (1 + t^i + t^{2i} + \cdots + t^{(p-1)i})^{d_i} = \sum_{n=0}^{s} c_n t^n,$$

where t is an indeterminate, $s = (p-1) \sum_{n=1}^{l} n d_n$, and, for $n \leq l$, $c_n = \dim_{\mathscr{K}} \overline{\mathscr{I}}^n / \overline{\mathscr{I}}^{n+1} = d^n$, as seen in 1.8.

On the other hand, c_n clearly depends only on the first n factors on the left-hand side: it is then possible to take limits on both sides as $l \to \infty$, to get

$$\prod_{i=1}^{\infty} (1 + t^i + \cdots + t^{(p-1)i})^{d_i} = \sum_{n=0}^{\infty} (dt)^n$$

and

$$\prod_{i=1}^{\infty} \left(\frac{1 - t^{ip}}{1 - t^i} \right)^{d_i} = \frac{1}{1 - dt}.$$

Taking logarithms on both sides, we obtain

$$\sum_{i=1}^{\infty} d_i \left(\sum_{j=1}^{\infty} \frac{t^{ij}}{j} - \sum_{j=1}^{\infty} \frac{t^{ijp}}{j} \right) = \sum_{j=1}^{\infty} \frac{(dt)^j}{j},$$

and comparing the coefficients of t^k we get

$$\sum_{ij=k} \frac{d_i}{j} - \sum_{ijp=k} \frac{d_i}{j} = \frac{d^k}{k},$$

or, equivalently

$$k \left(\sum_{ij=k} \frac{d_i}{j} - \sum_{ijp=k} \frac{d_i}{j} \right) = d^k,$$

whence

$$\sum_{i|k} i d_i - \sum_{i|k} i d_{i/p} = d^k \quad \text{and} \quad \sum_{i|k} i(d_i - d_{i/p}) = d^k.$$

Applying the Möbius inversion formula, we conclude that

$$d_n - d_{n/p} = \frac{1}{n}\sum_{k|n} \mu(k) d^{n/k}. \quad \square$$

Information about the power maps in F is now given. Let $\mu_s \colon F \to F$, $x \mapsto x^{p^s}$. We claim that, for each $l \geq 1$, a map

$$\overline{\mu}_s \colon \kappa_l/\kappa_{l+1} \to \kappa_{p^s l}/\kappa_{p^s l+1}$$

is induced: set $x = u_l u_{l+1}$, with $u_l \in \kappa_l$ and $u_{l+1} \in \kappa_{l+1}$. Then, via [H, Theorem III.9.4; HB, Proposition VIII.1.13], we have

$$u_l^{p^s} \equiv (u_l u_{l+1})^{p_s} \quad \text{modulo } \kappa_{p^s l+1}$$

(since $p^{s-r} | \binom{p^s}{p^r k}$ for $(k, p) = 1$, as in [HB, VIII.1.1(a), Proof]), and μ_s is well defined (see also, for instance, [L]). Clearly, $\overline{\mu}_s$ is not a homomorphism of groups, but

1.10 **Lemma.** $\overline{\mu}_s(x) = 1$ *if and only if* $x = 1$.

Proof. As seen in 1.6, F/κ_{lp^s+1} can be faithfully represented in the multiplicative group of $\mathscr{A}_{lp^s}$ by a homomorphism $\overline{\theta}$ setting

$$\overline{\theta}(x_i \kappa_{lp^s+1}) = 1 + a_i.$$

Using the notation of 1.6–1.8 and Jennings' theorem, we obtain

$$x \in \kappa_l \Leftrightarrow \overline{\theta}(x) - 1 \in \mathscr{B}^l.$$

Therefore it is enough to show, in $\mathscr{A}_{lp^s}$, that if $u \in \mathscr{B}^l$ and $(1+u)^{p^s} = 1$, then $u \in \mathscr{B}^{l+1}$.

To see this, suppose $u = u_1 + u_2$, where u_1 is a homogeneous polynomial of degree l and $u_2 \in \mathscr{B}^{l+1}$. Clearly,

$$1 = (1+u)^{p^s} = 1 + u^{p^s} = 1 + (u_1 + u_2)^{p^s} = 1 + u_1^{p^s},$$

and then $u_1^{p^s} = 0$.

Assume $u_1 = \lambda_1 w_1 + \cdots + \lambda_t w_t$, where w_i are distinct monic monomials of degree l, and $0 \neq \lambda_i \in k$. Now the distinct monomials of degree lp^s obtained juxtaposing p^s of the w_i to each other are linearly independent and $u_1^{p^s}$ is a linear combination of them.

It is now clear that in order to get $u_1^{p^s} = 0$ we must have $u_1 = 0$. $\square$

In the following three technical results we mention the basic commutators, and the commutator collecting process. For a definition of these concepts the reader is referred to [Ha, Chapter 11], or to the beginning of §2 of this paper.

1.11 **Lemma.** *Assume the notation as above. Let* $b_1 \cdots b_r$ *be a set of representatives of a basis of* κ_l/κ_{l+1}; *let* $c_1 \cdots c_t$ *be the basic commutators of weight* pl. *Then* $b_1^p \cdots b_r^p$, $c_1, \dots, c_t$ *are a set of representatives of a basis* $\kappa_{pl}/\kappa_{pl+1}$.

Proof. By 1.9, since [HB, Exercise VIII.21, p. 384]

$$t = \frac{1}{pl} \sum_{s|pl} \mu(s) d^{pl/s},$$

it is enough to show

$$\langle b_1^p, \dots, b_r^p, c_1, \dots, c_t, \kappa_{pl+1} \rangle \geq \kappa_{pl}.$$

By [HB, Proposition VIII.1.13],

$$\kappa_{pl} = [\kappa_{pl-1}, F](\kappa_l)^p.$$

By 1.5, and [Ha, Theorem II.2.4],

$$[\kappa_{pl-1}, F] \leq F_{pl}\kappa_{pl+1} = \langle c_1, \dots, c_t, \kappa_{pl+1} \rangle.$$

Furthermore, by the commutator collecting process

$$\kappa_l^p \subseteq \langle b_1^p, b_2^p, \dots, b_r^p, \kappa_{pl+1}, (\kappa_l)_p \rangle,$$

and by 1.5, $(\kappa_l)_p \leq F_{lp}\kappa_{pl+1}$. □

1.12 **Lemma.** *We keep the notation used so far. If* $p \nmid l$, *then* $\kappa_l = F_l\kappa_{l+1}$, *and the basic commutators of weight* l *are a set of representatives of a basis of* κ_l/κ_{l+1}.

Proof. By [HB, Proposition VIII.1.13], $\kappa_l = [\kappa_{l-1}, F]\kappa_m^p$, where m is the least integer such that $pm \geq l$; since $p \nmid l$, $\kappa_m^p \leq \kappa_{l+1}$ and $\kappa_l = [\kappa_{l-1}, F]\kappa_{l+1}$. By 1.4, $[\kappa_{l-1}, F] \leq F_l\kappa_{l+1}$. Therefore $\kappa_l = F_l\kappa_{l+1}$. Now 1.9 and [Ha, Theorem II.2.4] give the result. □

1.13 **Lemma.** *Let* F *and* κ_l *be as above. Suppose that* s, m, n *are integers,* p *a prime, such that* $\lceil n/p \rceil \leq s \leq m \leq n$. *Then* $(F_s\kappa_n) \cap \kappa_m = F_m\kappa_n$.

Proof. The inclusion $\supset$ is trivial. Set $\kappa_n = K$, and evaluate $[F_jK : F_{j+1}K]$ for $j = s, \dots, n-1$. Since $F_j^p \leq K$,

$$\frac{F_jK}{F_{j+1}K} \simeq \frac{F_j}{(F_j \cap F_{j+1}K)} = \frac{F_j}{F_j \cap F_{j+1}F_j^pK} = \frac{F_j}{F_{j+1}F_j^p(F_j \cap K)}$$

and this last is a factor of $F_j/F_{j+1}F_j^p$. Then

$$[F_jK : F_{j+1}K] \leq [F_j : F_{j+1}F_j^p] \leq p^{t_j},$$

where $t_j = \frac{1}{j}\sum_{k|j} \mu(k)d^{j/k}$ is the number of basic commutators of weight j, as in the proof of 1.11.

On the other hand, $F_{j+1}K \leq \kappa_{j+1}$, and by 1.11 and 1.12,

$$\begin{aligned} p^{t_j} &= [F_j\kappa_{j+1} : \kappa_{j+1}] = [F_jK : F_jK \cap \kappa_{j+1}] \\ &= [F_jK : (F_j \cap \kappa_{j+1})K] \leq [F_jK : F_{j+1}K]. \end{aligned}$$

All inequalities are then equalities, and $p^{t_j} = [F_jK : F_{j+1}K]$.

Similarly, we get $p^{t_j} = [F_j\kappa_m : F_{j+1}\kappa_m)$ for $j = s, \dots, m-1$. Now

$$[F_sK : F_sK \cap \kappa_m] = [F_s\kappa_m : \kappa_m] = p^{\sum_{j=s}^{m-1} t_j} = [F_sK : F_mK]. \quad \square$$

1.14 **Lemma.** *Assume* $x \in \kappa_{p^{n-1}+1} - \kappa_{p^n+1}$ *for* $n \geq 1$. *Then*

$$x^p \in \kappa_{p^n+1} - (\kappa_{p^n+1})'\kappa_{p^{n+1}+1}.$$

Proof. We first compute $(\kappa_{p^n+1})'$ modulo $\kappa_{p^{n+1}+1}$. We have

$$\begin{aligned} [\kappa_{p^n+1}, \kappa_{p^n+1}] &= \left[F^{p^{n+1}} \prod_{i=0}^{n} F_{p^i+1}^{p^{n-i}}, F^{p^{n+1}} \prod_{i=0}^{n} F_{p^i+1}^{p^{n-i}}\right] \\ &\leq [F^{p^{n+1}}, F^{p^{n+1}}] \prod_{i=0}^{n} [F^{p^{n+1}}, F_{p^i+1}^{p^{n-i}}] \prod_{i,j=0}^{n} [F_{p^i+1}^{p^{n-i}}, F_{p^j+1}^{p^{n-j}}]. \end{aligned}$$

The first factor, as well as each factor of the first product, is congruent to the identity, modulo $\kappa_{p^{n+1}+1}$; we now consider a factor of the second product, and evaluate it via 1.2. We have

$$(*) \qquad [F_{p^i+1}^{p^{n-i}}, F_{p^j+1}^{p^{n-j}}] \leq \prod F_{p^h(p^i+1)+p^k(p^j+1)}^{p^{2n-i-j-h-k}}$$

for all pairs (h, k) such that $0 \leq h \leq n-i$, $0 \leq k \leq n-j$. Now

$$F_{p^h(p^i+1)+p^k(p^j+1)}^{p^{2n-i-j-h-k}} \leq \kappa_{p^{2n-i-j-h-k}(p^h(p^i+1)+p^k(p^j+1))}.$$

Assume that either $h + i < n$ or $k + j < n$. Then

$$p^{2n-i-j-h-k}(p^h(p^i+1) + p^k(p^j+1)) \geq p^{n+1} + 1$$

and, modulo $\kappa_{p^{n+1}+1}$, we have $F_{p^h(p^i+1)+p^k(p^j+1)}^{p^{2n-i-j-h-k}} = 1$. Hence we may assume $h = n-i$, $k = n-j$, and the corresponding factor in $(*)$ is $F_{p^{n-i}(p^i+1)+p^{n-j}(p^j+1)} \leq F_{p^n+1}$.

Thus

$$(**) \qquad (\kappa_{p^n+1})' \leq F_{p^n+1} \quad \text{modulo } \kappa_{p^{n+1}+1}.$$

Assume now $x \in \kappa_j - \kappa_{j+1}$ for $p^{n-1} + 1 \leq j \leq p^n$. Hence $x^p \in \kappa_{pj} - \kappa_{pj+1}$, by 1.10, and, in particular, $x \in \kappa_{p^n+1} - \kappa_{p^{n+1}+1}$. Now we compute modulo $\kappa_{p^{n+1}+1}$. By 1.13, we get $\kappa_{pj} \cap F_{p^n+1} = F_{pj}$, and by 1.11, $x^p \notin F_{pj}$; thus $x^p \notin F_{p^n+1}$. But $(**)$ gives now $x^p \notin (\kappa_{p^n+1})'$. $\square$

We now come to:

Proof of the main theorem. We will show the Theorem for $X = X(d, r) = F/\kappa_{p^r+1}(F)$, where F is a free group on a finite number d of generators, $r \geq 0$ is an integer, and $\kappa_{p^r+1}(F)$ is defined as in 1.3; since we have noted in the introduction that homomorphic images of FW-complements are FW-complements, and every p-group is a homomorphic image of $X(d, r)$ for a suitable choice of d and r, we will have our result. We indicate the subgroup $\kappa_m(F)$ simply by κ_m, for every positive integer m.

We begin with the following remarks:

1. Every element of $F - \kappa_{p^r+1}$ has a power in $\kappa_{p^{r-1}+1} - \kappa_{p^r+1}$.

Let $x \in F - \kappa_{p^r+1}$. Then there exists h, $1 \leq h \leq p^r$, such that $x \in \kappa_h - \kappa_{h+1}$. Let s be maximal such that $p^s h \leq p^r$. We have $p^s h \geq p^{r-1}$, or s would not be maximal. By 1.10, $x^{p^s} \in \kappa_{p^s h} - \kappa_{p^s h+1}$. Thus $x^{p^s} \in \kappa_{p^{r-1}+1} - \kappa_{p^r+1}$.

2. Every element of $F - \kappa_{p^r+1}$ has a power in $\kappa_{p^r+1} - (\kappa_{p^r+1})'\kappa_{p^{r+1}+1}$.

This is straightforward by (1) and 1.14.

3. $\kappa_{p^r+1}/(\kappa_{p^r+1})'\kappa_{p^{r+1}+1}$ is elementary abelian.

It is enough to show that $\kappa_{p^r+1}/\kappa_{p^{r+1}+1}$ has exponent p. But this is clear, by [HB, VIII.1.13(b)].

Now κ_{p^r+1} is a free group, by Schreier's Theorem, and, since it has finite index in F, κ_{p^r+1} is finitely generated. Let m be its rank. Clearly, m can be computed using Schreier's Formula and 1.9. We choose t such that $p^t > m$, set

$$\overline{F} = F/\kappa_{p^t+1}(\kappa_{p^r+1}(F)),$$

and use bar-notation consistently. We note here that $\overline{F}$ is a finite p-group.

Since, by (3), $\overline{\kappa_{p^r+1}/(\kappa_{p^r+1})'\kappa_{p^{r+1}+1}}$ is elementary abelian, we have

$$(*) \qquad \overline{(\kappa_{p^r+1})'\kappa_{p^{r+1}+1}} \geq \Phi(\overline{\kappa_{p^r+1}}).$$

Now, by 1.6, we can apply Wall's Theorem 1.7 to $G = \overline{\kappa_{p^r+1}}$, and $q = p^t$; we get that every element in $\overline{\kappa_{p^r+1}} - \Phi(\overline{\kappa_{p^r+1}})$ has a power in $(\overline{\kappa_{p^r+1}})^{p^t} - K$, where K is a suitable maximal subgroup of $(\overline{\kappa_{p^r+1}})^{p^t}$.

Now $(*)$, together with (2), shows that an element of $\overline{F} - \overline{\kappa_{p^r+1}}$ has a power in $(\overline{\kappa_{p^r+1}})^{p^t} - K$, and, by 1.1, we have that $\overline{F}/\overline{\kappa_{p^r+1}} \cong F/\kappa_{p^r+1} \cong X$ is an FW-complement for $\overline{F}$. $\square$

2

We begin this section with some known material. Here, p will be an odd prime.

2.1 **Definition.** Let F be a free group, freely generated by the elements x_1, $\ldots$, x_d. The *basic commutators* on $x_1, \ldots, x_d$ are the elements of the ordered infinite set $\{c_i\}_{i \in \mathbf{N}}$ defined inductively as follows [Ha, p. 178]:

(1) $c_i = x_i$, for $i \leq d$, are the basic commutators of weight 1.

(2) Suppose we have defined the well-ordered set $c_1, \dots, c_r$ of the basic commutators of weight less than n. Those of weight n will be the commutators $[u, v]$ such that

(a) u, v are basic commutators, and the sum of their weights is n,
(b) $u > v$,
(c) if $u = [w, t]$, then $v \geq t$.

Furthermore, the basic commutators of weight n will follow all those of lower weight, and if $[u_1, v_1]$, $[u_2, v_2]$ have weight n, we will have $[u_1, v_1] < [u_2, v_2]$ if either $v_2 > v_1$ or $v_2 = v_1$ and $u_2 > u_1$.

We now state P. Hall's theorem about basic commutators (see [Ha] as a reference).

2.2 **Theorem.** *We may collect the product $(x_1 \cdots x_d)^n$ in the form*

$$(x_1 \cdots x_d)^n = c_1^n c_2^n \cdots c_d^n (c_{d+1})^{k_{d+1}} \cdots c_r^{k_r} t_1 \cdots t_s,$$

where the c_i's are the basic commutators in $x_1, \dots, x_d$ in order, and $t_1, \dots, t_s$ are basic commutators later than c_r in the ordering. For $d+1 \leq i \leq r$, the element k_i is of the form $k_i = \sum_{j=1}^{m} b_j \binom{n}{j}$, if m is the weight of c_i; b_j does not depend on n but only on c_i.

We will not give the proof of Theorem 2.2 here, but refer the reader to [Ha, Theorem 12.3.1]. The proof includes a method for the explicit computation of the b_j's for each basic commutator c_i, via the introduction of a suitable ordered set Λ_i depending only on c_i.

The following example, which is relevant in the rest of this section, is taken from some unpublished lecture notes of a course taught by J. Alperin at the University of Chicago in the Summer Quarter 1984.

Set $d = 2$, $x_1 = x$, $x_2 = y$, and let p be an odd prime. Let

$$c(i) = [[y, x; p-i], y; i-1].$$

As in [Ha, pp. 180–181], the coefficient b_j for $c(i)$ is given by the number of order preserving maps of the set $\Lambda = \{\lambda_1 \cdots \lambda_p\}$ with order given by the inequalities

$$\lambda_{p-i+1} > \lambda_{p-1} > \cdots > \lambda_2 > \lambda_1 < \lambda_{p-i+2} < \cdots < \lambda_p$$

onto the set $\{1, 2, \dots, j\}$ with the natural order.

Our goal is now to compute, modulo p, the exponent $k(i)$ of $c(i)$ in the expansion of $(xy)^p$. We have $k(i) = \sum_{j=1}^{p} b_j \binom{p}{j}$. All the summands are now divisible by p, except the last one, which is b_p. We have to compute the number of all possible order preserving surjective maps $\psi: \Lambda \to \{1, 2, \dots, p\}$. Clearly, we must have $\psi(\lambda_1) = 1$, since ψ is surjective and $\lambda_1 < \lambda_l$, $l = 2, \dots, p$. Clearly a choice of $\psi(\lambda_2) < \cdots < \psi(\lambda_{p-i+1})$ in the set $\{2, 3, \dots, p\}$ determines ψ. Therefore, there are $\binom{p-1}{p-i}$ such functions. We have proved

2.3 **Lemma.** $k(i) \equiv \binom{p-1}{p-i} \bmod p$.

The expansions of powers of products given by 2.2 for free groups hold in fact in every group: let G be a group, let $g_1, \dots, g_d$ be any elements of G. Let F be as in 2.1, and let $\varphi: F \to G$ be the homomorphism determined by the assignment $\varphi(x_i) = g_i$. Then we can apply φ to both sides of the expression in 2.2, and get an expansion for $(g_1 \cdots g_d)^n$. In what follows, we will refer to this last expansion as the one obtained "via the commutator collecting process".

We can now prove

2.4 **Lemma.** *Let G be a group, and assume $a \in G_n$ and $b \in G_m$. Assume $m \geq n$. Then*

$$[a^p, b] \equiv [b, a; p]^{-1} \quad \bmod G_{pn+m+1}\kappa_{p(n+m)}(G).$$

Proof. We have

$$(1) \qquad [a^b, a^{-1}] = [a, b, a^{-1}].$$

Now we compute via the commutator collecting process, and all the congruences will be $\bmod N = G_{pn+m+1}\kappa_{p(n+m)}(G)$. We have

$$1 \equiv [a, b]^p \equiv a^{-p}(a^b)^p \prod_{i=1}^{p-1} \overline{c}(i)^{k(i)},$$

where $k(i)$ is as in 2.3 and

$$\overline{c}(i) = [[a^b, a^{-1}; p-i], a^b; i-1],$$

since the pth powers of commutators of weight less than p in a^b, a^{-1} are in $\kappa_{p(m+n)}(G)$, by (1), and the other commutators of weight greater than or equal to p in a^b, a^{-1} appearing in the expansion are in G_{pn+m+1}, again by (1).

Also, $\overline{c}_i$ has order $p \bmod N$, and then, by 2.3, we may write $\binom{p-1}{p-i}$ for k_i. Hence we get, $\bmod N$

$$1 \equiv a^{-p}(a^b)^p \prod_{i=1}^{p-1} \overline{c}_i^{\binom{p-1}{p-i}} = [a^p, b] \prod_{i=1}^{p-1} \overline{c}_i^{\binom{p-1}{p-i}}.$$

But

$$\overline{c}_i \equiv [[a, b, a^{-1}; p-i], a; i-1] \equiv [b, a; p]^{(-1)^{p-i+1}} \qquad \bmod N$$

by (1) and [H, Proposition III.6.8]. Then

$$1 \equiv [a^p, b][b, a; p]^{\sum_{i=1}^{p-1}(-1)^{p-i+1}\binom{p-1}{p-i}} = [a^p, b][b, a; p],$$

and the result follows. □

From now on, F will be the free group on two generators x and y. We set $\mathscr{P} = \{y^p, (xy^i)^p, \ i = 0, 1, \dots, p-1\}$. We recall that a left-normed commutator in x, y is one of the form $[z_1, \dots, z_n]$ where $z_i = x$ or $z_i = y$.

2.5 **Proposition.** *The elements of $\mathscr{P}$ form, with the non-left-normed basic commutators of weight p, a set of representatives for a basis of $\kappa_p(F)/\kappa_{p+1}(F)$.*

Proof. Let $\mathscr{L}$ be the set of the left-normed basic commutators of weight p, and $\mathscr{N}$ be the set of non-left-normed basic commutators of weight p. By 1.11, $\mathscr{B} = \{x^p, y^p\} \cup \mathscr{L} \cup \mathscr{N}$ is a set of representatives of a basis of $\kappa_p(F)/\kappa_{p+1}(F)$. Note that $|\mathscr{L}| = p-1$, so that $|\mathscr{P} \cup \mathscr{N}| = |\mathscr{B}|$. We set $N = \langle \mathscr{N}, \kappa_{p+1}(F)\rangle$, $G = F/N$.

Since $\mathscr{N} \subseteq \mathscr{B}$, our claim is proved if we show that the elements of $\mathscr{P}$ represent a set of generators of $K_p(G)$. Thus we argue modulo N.

Via the commutator collecting process, one obtains

$$(xy)^p \equiv x^p y^p c_1^{k_1} \cdots c_{p-1}^{k_{p-1}} \tag{1}$$

modulo N, where $c_l = [[y, x; p-l], y; l-1]$; and since the nontrivial elements of G_p have order $p \pmod N$, we may assume by 2.3 that

$$k_l = \binom{p-1}{p-l}. \tag{2}$$

Then (1) becomes

$$y^{-p}x^{-p}(xy)^{-p} = \prod_j c_j^{\binom{p-1}{p-j}}. \tag{3}$$

Now, for each i such that $1 \le i \le p-1$, setting $y^{-ip}x^{-p}(xy^i)^p = u_i$, substituting y^i for y in (3), and switching to additive notation, since $G_{p+1} = 1$, one obtains from (3) a system of $p-1$ linear equations:

$$u_i = \sum_j \binom{p-1}{p-j} i^j c_j. \tag{4}$$

The matrix

$$\left(\binom{p-1}{p-j} i^j\right) \tag{5}$$

is the product of a nonsingular diagonal matrix on $\mathbf{Z}_p$ and a nonsingular Vandermonde matrix; therefore (5) is nonsingular, and it can be inverted. The c_j can then be expressed as linear combinations of the u_i's. Then $x^p, y^p, u_1, \dots, u_{p-1}$ are (a set of representatives of) a basis of $\kappa_p(G)$, and since $(xy^i)^p = x^p y^{ip} u_i$, the proof is complete. □

Since in a metabelian group all non-left-normed basic commutators are trivial, we have, as a corollary, a well-known result by Meier-Wunderli: metabelian p-groups of exponent p on two generators have class at most $p-1$. We have also:

2.6 **Corollary.** *Set* $G = F/\kappa_{p^2+1}(F)$. *Then* $G^{p^2} \cap G_{p+1} = 1$.

Proof. $G^{p^2} \cap G_{p+1} \le \kappa_{p^2}(G) \cap G_{p+1} = G_{p^2}$, by 1.13. By 1.11, a basis of $\kappa_{p^2}(G)$ is given by the basic commutators of weight p^2 and the pth powers of the elements that represent a basis of $\kappa_p(G)/\kappa_{p+1}(G)$. The former generate G_{p^2}, while G^{p^2} is contained in the subgroup generated by the latter, by 2.5. Then $G^{p^2} \cap G_{p^2} = 1$. □

The following is a corollary to 1.5:

2.7 **Lemma.** *Let* $G, p, m_i, n, r, \kappa_{m_i}$ *be as in* 1.5. *Let* $x_i \in \kappa_{m_i}$, $i = 1, \dots, s$, *and, for a given* $k \le s$, *let* $y_k \in \kappa_{m_k}$. *Then*

$$\begin{aligned}&[x_1, \dots, x_{k-1}, x_k y_k, x_{k+1}, \dots, x_s]\\ &\qquad \equiv [x_1, \dots, x_s][x_1, \dots, x_{k-1}, y_k, x_{k+1}, \dots, x_s] \pmod{G_{r+1}\kappa_{n+1}(G)}.\end{aligned}$$

Proof. The proof is by induction on s. The result is immediate for $s = 1$. Assume the result true for $s - 1$, and, without loss of generality, assume

$$G_{r+1}\kappa_{n+1}(G) = 1. \tag{1}$$

To begin, suppose $k = s$. Since

$$\begin{aligned}&[x_1, \dots, x_{s-1}, x_s y_s]\\ &\qquad = [x_1, \dots, x_{s-1}, y_s][x_1, \dots, x_{s-1}, x_s][x_1, \dots, x_{s-1}, x_s, y_s]\end{aligned}$$

and since, by 1.5,

$$[x_1, \dots, x_{s-1}, y_s] \in G_r\kappa_n(G), \qquad [x_1, \dots, x_s] \in G_r\kappa_n(G),$$

the result follows by (1) and [HB, Proposition VIII.1.13]. Now assume $k < s$. By induction,

$$\begin{aligned}&[x_1, \dots, x_k, \dots, x_{s-1}][x_1, \dots, x_{k-1}, y_k, x_{k+1}, \dots, x_{s-1}]\\ &\qquad = [x_1 \cdots x_{k-1}, x_k y_k, x_{k+1}, \dots, x_{s-1}]u,\end{aligned}$$

where $u \in G_{\hat{m}_s+1}\kappa_{p(r-m_s)+1}(G)$; here $r = \min_{1\le i\le s-1}(\hat{m}_i + \lceil m_i/p \rceil)$. By 1.5 and (1), to prove the result it is enough to show $[u, x_s] = 1$. Set $u = u_1u_2$, where $u_1 \in G_{\hat{m}_s+1}$ and $u_2 \in \kappa_{p(r-m_s)+1}(G)$. Then

$$[u, x_s] = [u_1u_2, x_s] = [u_1, x_s][u_1, x_s, u_2][u_2, x_s].$$

Thus, it is enough to show that $[u_1, x_s]$ and $[u_2, x_s]$ are trivial.

We have $[u_1, x_s] \in [G_{\hat{m}_s+1}, \kappa_{m_s}]$. Now κ_{m_s} is the product of the subgroups of the form $(G_v)^{p^w}$, with $vp^w \ge m_s$. Therefore, in order to get $[u_1, x_s] = 1$, it is enough to show that $[G_{\hat{m}_s+1}, G_v^{p^w}] = 1$, with v, w as above. By 1.4, and assumption (1), we have

$$[G_{\hat{m}_s+1}, G_v^{p^w}] \le G_{\hat{m}_s+1+vp^w}\kappa_{p(\hat{m}_s+1+vp^{w-1})} \le G_{r+1}\kappa_{p(\hat{m}_s+1+\lceil m_s/p\rceil)} \le \kappa_{p(\lceil n/p\rceil+1)} = 1.$$

Now we want $[u_2, x_s] = 1$; but $[u_2, x_s] \in [\kappa_{(r-m_s)p+1}(G), \kappa_{m_s}(G)]$. Therefore it is enough to show

$$[\kappa_{(r-m_s)p+1}(G), \kappa_{m_s}(G)] = 1.$$

We want to apply 1.5; in order to do so, we first note that

$$\left\lceil \frac{(r-m_s)p+1}{p} \right\rceil + m_s = r - m_s + 1 + m_s = r + 1 \geq \left\lceil \frac{n+1}{p} \right\rceil$$

since $r \geq \lceil n/p \rceil$, and that

$$(2) \qquad (r-m_s)p \geq \sum_{i=1}^{s-1} m_i,$$

since, for some j, we have $r = \hat{m}_j + \lceil m_j/p \rceil$ and

$$(r-m_s)p = \left(\left\lceil \frac{m_j}{p} \right\rceil + \sum_{i \neq j, s} m_i \right) p = \left\lceil \frac{m_j}{p} \right\rceil p + \sum_{i \neq j, s} pm_i$$

$$\geq m_j + \sum_{i \neq j, s} pm_i \geq \sum_{i=1}^{s-1} m_i.$$

Hence

$$(r-m_s)p + 1 + \left\lceil \frac{m_s}{p} \right\rceil \geq \sum_{i=1}^{s-1} m_i + \left\lceil \frac{m_s}{p} \right\rceil + 1 \geq \left\lceil \frac{n}{p} \right\rceil + 1 \geq \left\lceil \frac{n+1}{p} \right\rceil,$$

by hypothesis. Now 1.5 gives

$$[\kappa_{(r-m_s)p+1}(G), \kappa_{m_s}(G)] \leq G_{(r-m_s)p+m_s+1}\kappa_{n+1}(G).$$

By (2),

$$(r-m_s)p + m_s + 1 \geq \sum_{i=1}^{s} m_i + 1;$$

by (1) we have the result. □

The following result is an easy consequence of Lemma 2.7 in [D].

2.8 **Lemma.** *Let F be the free group on two generators x and y. Let p be an odd prime, m an integer, $m \geq 2$. Let $c_1, c_2, \ldots, c_r$ be the left-normed basic commutators in x, y of weights ranging from 2 to $p-1$. Let c be any commutator of total weight $n \leq m$, and assume c is a commutator in $c_1, c_2, \ldots, c_r$. Let B_n be the family of basic commutators in x, y of weight not less than n that are also commutators in $c_1, c_2, \ldots, c_r$. Then c can be written, modulo $\kappa_{m+1}(F)$, as a product of elements of B_n.*

2.9 **Proposition.** *Let p be a prime, $p \geq 5$. Then there exists a finite p-group G, and a complex module M for G, such that $G/N(G, M)$ is extra-special of order p^3 and exponent p.*

Proof. Let F be the free group on two generators x and y, and set $G = F/\kappa_{p^2+1}(F)F_{3p-1}$, $N = \kappa_3(G)$, and $H = \kappa_{2p}(G)$. We show first that there exists a subgroup K of H such that G, N, H, and K satisfy condition (ii) of 1.1. Then, if M is the complex module induced as in 1.1 from a linear module of H having K as the kernel, we will show that $N = N(G, M)$. Since it is easy to show that G is a finite p-group and G/N is extra-special of order p^3 and exponent p, we will have the result. We divide the proof into steps; we indicate elements of G by their representatives in F, without possibility of confusion.

Step 1. *H is elementary abelian, and $H = H_1 \oplus H_2 \oplus H_3 \oplus H_4 \oplus H_5 \oplus H_6$, where $H_i = \langle \mathscr{B}_i \rangle$, and*

$\mathscr{B}_1 = \{[y, x]^p\}$,
$\mathscr{B}_2 = \{p$th powers of basic commutators in x, y of weight w, $3 \le w \le p-1\}$,
$\mathscr{B}_3 = \{p$th powers of non-left-normed basic commutators of weight $p\}$,
$\mathscr{B}_4 = \{p$th powers of elements of $\mathscr{P}\}$,
$\mathscr{B}_5 = \{$basic commutators of weight w, $2p \le w \le 3p-3\}$,
$\mathscr{B}_6 = \{$basic commutators of weight $3p-2\}$.

By [HB, Proposition VIII.1.13], $(\kappa_{2p}(G))^p \le \kappa_{2p^2}(G) = 1$ and therefore H has exponent p. By definition, $H = G^{p^2}(G_2)^p G_{2p}$. Therefore, since $G^{p^2} \le Z(G)$, we have

$$[H, H] = [G_2^p G_{2p}, G_2^p G_{2p}] \le [G_2^p, G_2^p][G_{2p}, G_2^p] G_{4p}.$$

But $G_{4p} = 1$, since G has class $3p-2$; by 1.2

$$[G_2^p, G_2^p] \le G_{4p}(G_{2+2p})^p (G_4)^{p^2} = 1$$

and

$$[G_{2p}, G_2^p] \le G_{4p}(G_{2p+2})^p = 1.$$

Now we have shown that H is elementary abelian. By 1.11, 1.12, and 2.5, $\bigcup_{i=1}^6 \mathscr{B}_i$ is a basis for H. Then $H = \bigoplus_{i=1}^6 H_i$.

Step 2. *Every element in $G - N$ has a nontrivial power in H.* By 2.6, letting $\mathscr{G} = F/\kappa_{p^2+1}(F)$, we get $\mathscr{G}^{p^2} \cap \mathscr{G}_{3p-1} = 1$, while $\mathscr{G}/\mathscr{G}_{3p-1} = G$. By 1.10, every element in $\mathscr{G} - \phi(\mathscr{G})$ has a nontrivial p^2-power in $\mathscr{G}$; then also every element in $G - \phi(G)$ has a nontrivial p^2-power in G, and so clearly in H. Elements of $\kappa_2(G) - N$ have, similarly, their nontrivial pth powers in H, by 1.10.

Step 3. $[\kappa_{p+1}(G), \kappa_2(G); p-1] = 1$. In fact, 1.5 is applicable for $n = p^2 + 1$, $s = p$, $m_1 = p+1$, $m_2 = \cdots = m_p = 2$. Hence

$$[\kappa_{p+1}(G), \kappa_2(G); p-1] \le G_{3p-1}\kappa_{p^2+1}(G) = 1.$$

Step 4. *Let $v \in G^p \kappa_{p+1}(G)$. Then*

$$[v, [y, x]; p-1] = 1 \quad \textit{if and only if} \quad v \in \kappa_{p+1}(G).$$

The "if" part has been seen already in Step 3. By 1.2, we are now able to define a map

$$\psi\colon G^p\kappa_{p+1}(G)/\kappa_{p+1}(G) \to G_{3p-2}/G_{3p-1}$$
$$w \mapsto [w, [y, x]; p-1].$$

By 2.7, ψ is a homomorphism. It is enough to show that it is an isomorphism, and so to exhibit $p+1$ independent elements in $\operatorname{Im}\psi$. Now, by 2.5 and 1.11, x^p, y^p, and the left-normed basic commutators of weight p are a basis of

$$G^p\kappa_{p+1}(G)/\kappa_{p+1}(G),$$

and then it is enough to show that their images under ψ are powers of *distinct* basic commutators of weight $3p-2$.

Indeed, it is so, since

$$[x^p, [y, x]; p-1] = [[y, x; p+1], [y, x]; p-2]^{-1},$$
$$[y^p, [y, x]; p-1] = [[y, x, y; p], [y, x]; p-2]^{-1}$$

by 2.4 and repeated application of 2.7.

Set $\mathscr{B}_6' = \mathscr{B}_6'' \cup \mathscr{B}_6'''$, where

$$\mathscr{B}_6'' = \{\text{all basic commutators of weight } 3p-2, \text{ except those in the image of } \psi \text{ as in Step 4}\},$$
$$\mathscr{B}_6''' = \{v^p\psi(v) | v \in \mathscr{P}\}.$$

Set also $H_6' = \langle \mathscr{B}_6' \rangle$. (Warning: in spite of the notation, H_6' is *not* the derived subgroup of H_6!) We have

$$H = H_1 \oplus H_2 \oplus H_3 \oplus H_4 \oplus H_5 \oplus H_6'$$

since H_6' is a complement for $\bigoplus_{i=1}^5 H_i$ in H. Set

$$K_0 = H_2 \oplus H_3 \oplus H_5 \oplus H_6'.$$

Step 5. *Computation of g^p, with $g \in \kappa_2(G) - N$.* Suppose $g \in \kappa_2(G) - N$, and assume $g = a_2 a_{p+1}$, where $a_{p+1} \in \kappa_{p+1}(G)$ and $a_2 \in \kappa_2(G)$. Then, since the pth powers of a_{p+1} and of commutators of weight less than p in a_{p+1}, a_2 are in κ_{p^2+1} and the commutators of weight p or larger in a_{p+1}, a_2 are trivial by Step 3, we get $g^p = a_2^p$.

Hence, by 2.5, we may assume that $g \in \kappa_2(G) - N$ is a product $g = uv$, where u is a product of basic commutators of weights ranging from 2 to $p-1$ and of non-left-normed basic commutators of weight p, and v is a product of powers of the elements of $\mathscr{P}$. Again, expanding, we get $g^p = u^p v^p [v, u; p-1]$ since all pth powers of commutators in v and u are 1, as $v \in K_p(G)$, and among the other factors, only $[v, u; p-1] \in G_{3p-2}$ can be nontrivial, by 1.4. But $u = [y, x]^k \cdot u_3$, with $u_3 \in \kappa_3(G)$ for some integer k, $1 \le k \le p-1$. Expanding, by 2.7 and 1.4,

$$[v, u; p-1] = [v, [y, x], p-1]^{k^{p-1}} = [v, [y, x]; p-1].$$

Hence $g^p = u^p v^p[v, [y, x]; p-1]$. By Step 4, we have now that $g^p \equiv u^p$, modulo K_0, and $u = [y, x]^k w_1 w_2$, where w_1 is a product of basic commutators in x, y of weights ranging from 3 to $p-1$, and w_2 is a product of non-left-normed basic commutators of weight p. Set $u' = [y, x]^k w_1$, and expand. We get $g^p \equiv u^p = (u' w_2)^p \equiv (u')^p$, modulo K_0, since $w_2^p \in H_3$; all commutators in u' and w_2 have weight at least $p+2$ (as commutators in x, y), and therefore their pth powers are trivial; $[w_2, u; p-1] \in H_6'$, by 2.7 and 1.4, as w_2 is a product of non-left-normed basic commutators of weight p; and, finally, every other commutator appearing in the expansion is trivial, because of its total weight in x and y, and 1.4.

We continue, and expand $(u')^p = ([y, x]^k w_1)^p = ([y, x]^k c_1 c_2 \cdots c_r)^p$, where $c_1, c_2, \ldots, c_r$ are basic commutators of weight ranging from 3 to $p-1$. We note that $c_i^p \in H_3$, and the other commutators appearing in the expansion, via repeated applications of 2.8, are shown to be in $H_2 \oplus H_3 \oplus H_5 \oplus \langle \mathscr{B}_6'' \rangle \leq K_0$. All this adds up to $g^p \equiv [y, x]^{kp}$ modulo K_0.

Now we set $K = \langle K_0, bc^{-1} | b, c \in \mathscr{B}_1 \cup \mathscr{B}_4 \rangle$.

Step 6. *Every element of* $G-N$ *has a power in* $H-K$, *and* G, N, H, K *satisfy hypothesis* (ii) *in* 1.1. If $g \in G - \kappa_2(G)$, then $g = ab$, where $a \in \{x^i y^j | i, j = 0, 1, \ldots, p-1, (i, j) \neq (0, 0)\}$ and $b \in \kappa_2(G)$. Expanding, we conclude that the p^2th power of g is a nontrivial power of an element of $\mathscr{B}_4$. Furthermore, we have shown in Step 5 that if $g \in \kappa_2(G) - \kappa_3(G)$, then g^p is congruent to a nontrivial power of $[y, x]$ modulo K_0, and thus also modulo K. But K is maximal in H, and hypothesis (ii) of 1.1 is satisfied by G, N, H, K.

By 1.1, we have that $N \geq N(G, M)$, where M is induced from a complex linear module L of H such that the kernel of the representation afforded by L is K. We now complete the proof of 2.9 by showing

Step 7. $N = N(G, M)$. It is enough to show that $N(G, M) \geq N$. Certainly $N(G, M) \geq \kappa_{p+1}(G)$: in fact, $\kappa_{p+1}(G)$ has exponent p, by [HB, VIII.1.13(b)], and we can apply Corollary 1.5 in [S1]. Let c be a basic commutator of weight w, $3 \leq w \leq p-1$. Since $c^p \in H_2 \leq K$, we have that c^p is trivial on L, and then, as in the proof of 1.1, c fixes a nontrivial vector in M. Thus $c \in N(G, M)$. On the other hand, if $c = [a, b]$ is a basic commutator of weight p, a must be a basic commutator of weight 3 or more, since $p \geq 5$, and thus an element of $N(G, M)$. But $N(G, M)$ is normal, hence $c \in N(G, M)$. By equation (3) in the proof of 2.5, we can conclude that $x^{-p} y^{-p} (xy)^p \in N(G, M)$. We use the commutator collecting process once more to show that $(x^{-p}(xy)^p)^p = x^{-p^2}(xy)^{p^2} \in K$, and $x^{-p}(xy)^p \in N(G, M)$. Then $y^p \in N(G, M)$, and similarly $x^p \in N(G, M)$. Hence, by 1.11 and 1.12,

$$\begin{aligned} N &= \kappa_3(G) \\ &= \langle \kappa_{p+1}(G), x^p, y^p, c | c \text{ is a basic commutator of weight } w,\ 3 \leq w \leq p \rangle \\ &\leq N(G, M). \quad \square \end{aligned}$$

2.10 *Remark.* In [S2] a construction similar to the above is given in the $p = 3$ case, showing that there is a 3-group G with a normal subgroup N and a complex module M such that $N \geq N(G, M)$, and G/N is extra-special of order 27 and exponent 3.

However, in this case we do not have $N = N(G, M)$. In fact, $G/N(G, M)$ turns out to be isomorphic to the group of order 3^4 given, for $p = 3$, by [H, III.10.15]. Here is a sketch of a proof: as in [S2], let F be the free group on two generators x, y, let $p = 3$, and $G^* = F/\kappa_{10}(F)F_8$, $G = G^*/V'$, where V' is a suitable subgroup of $Z(G^*)$. From [S2] we know that the 3rd-power map establishes an isomorphism α between $\kappa_3(G)/\kappa_4(G)$ and $\kappa_9(G)$. We can choose $H = \kappa_9(G)G_2^3$, and K to be a suitable maximal subgroup of H which turns out to be normal in G. This is enough to conclude that $N(G, M)/\kappa_4(G)$ is the preimage under α of $K \cap \kappa_9(G)$, and since every element of $G - \Phi(G)$ has its 9th power outside $K \cap \kappa_9(G)$, and thus its third power outside $N(G, M)$, we have the result.

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